Two-Commodity Flow (2CF) is equivalent to Linear Programming (LP)

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Problem definition -- Linear Programs (LP)

Optimization problem with linear objective and linear constraints

Given

- a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$,
- a vector $\boldsymbol{b} \in \mathbb{R}^{m}$, a vector $\boldsymbol{c} \in \mathbb{R}^{n}$,

find $\boldsymbol{x} \in \mathbb{R}^{n}$ s.t.

$$\begin{aligned} \max \quad & \boldsymbol{c}^{\top}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

Polynomially-bounded assumption

1. magnitude: $\|\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}\|_{max} \le X$
2. polytope radius: $\|\boldsymbol{x}\|_1 \le R$

$$X, R = \mathrm{poly}(nnz(\boldsymbol{A}))$$

SOTA LP solver runtime [Cohen-Lee-Song'19, Jiang-Song-Weinstein-Zhang'20]

$$\tilde{O}\left(nnz(\boldsymbol{A})^{\max\{\omega,\, 2.055\}}\right), \quad \omega \approx 2.37$$

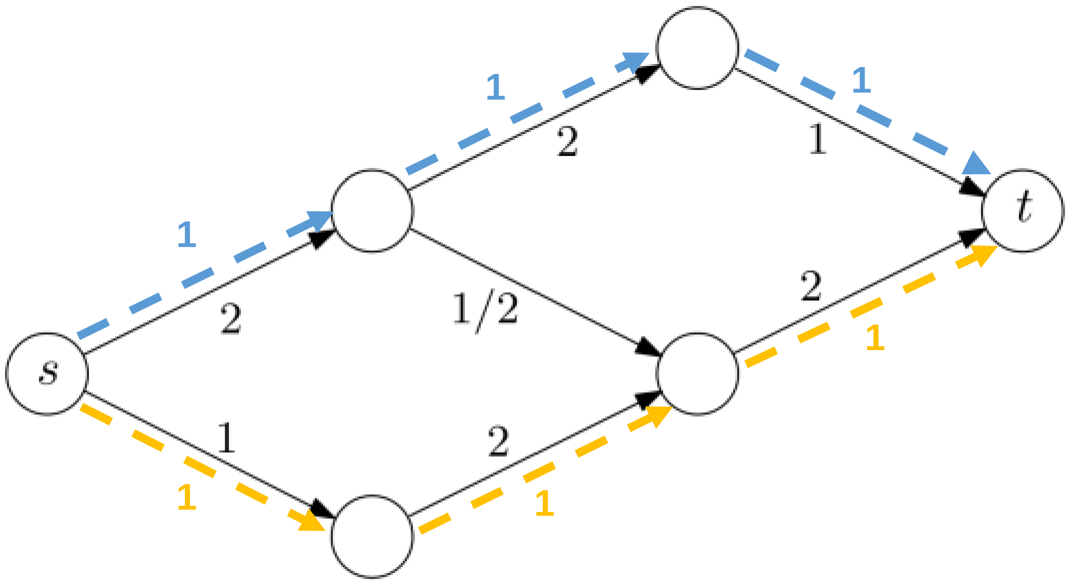
$\tilde{O}$ hides polylog dependences.

(Single-commodity) maxflow problem (1CF)

Given

- a directed graph $G = (V, E)$,
- a source-sink pair (s, t),
- edge capacity $\boldsymbol{u} \in \mathbb{R}_{\geq 0}^{|E|}$

Find a **feasible** flow $\boldsymbol{f} \in \mathbb{R}^{|E|}$ s.t. the amount of flow routed from s to t is maximized.



$maxFlow = 2$

SOTA maxflow solver runtime [Chen-Kyng-Liu-Peng-Probst-Sachdeva'22]

$$\tilde{O}\left(|E|^{1+o(1)}\right)$$

1CF as LP

- $\boldsymbol{f}$ is a feasible flow if it satisfies

1. **Flow conservation constraints**

$$\sum_{incoming} \boldsymbol{f}(\cdot, v) - \sum_{outgoing} \boldsymbol{f}(v, \cdot) = 0, \qquad \forall v \in V \backslash \{s, t\}$$

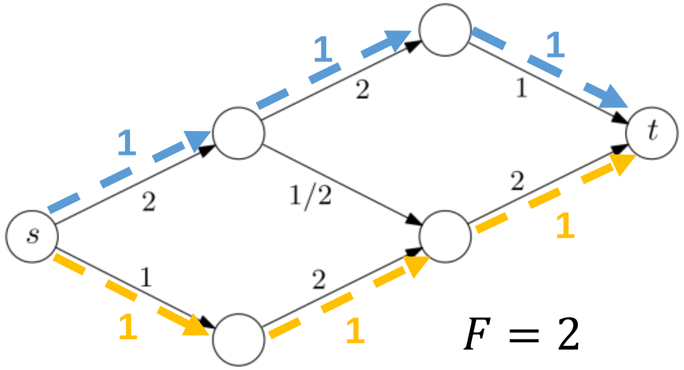
$$\sum_{outgoing} \boldsymbol{f}(s, \cdot) = \sum_{incoming} \boldsymbol{f}(\cdot, t) = F$$

2. **Capacity constraints**

$$\boldsymbol{f}(e) \leq \boldsymbol{u}(e), \qquad \forall\, e \in E$$

3. **Direction constraints**

$$\boldsymbol{f}(e) \geq 0, \qquad \forall\, e \in E$$



$$\begin{array}{ll} \max & F \\ \text{s.t.} & \boldsymbol{B}\boldsymbol{f} = F\boldsymbol{d} \\ & \boldsymbol{f} \leq \boldsymbol{u} \\ & \boldsymbol{f} \geq \boldsymbol{0} \end{array}$$

LP solver: $\tilde{O}\left(|E|^{\max\{\omega, 2.055\}}\right)$

(since $nnz = \Theta(|E|)$)

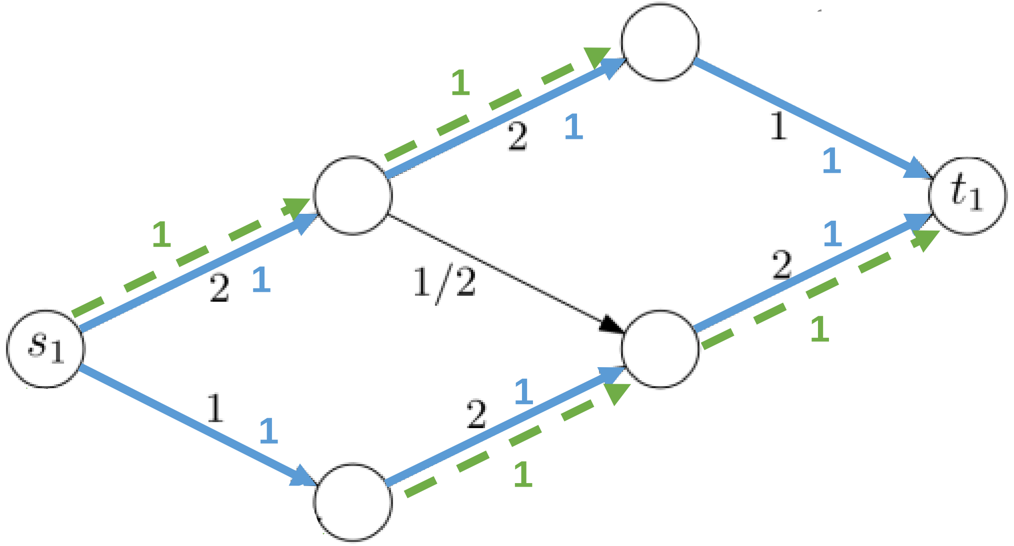
1CF solver: $\tilde{O}\left(|E|^{1+o(1)}\right)$

Two-commodity maxflow problem (2CF)

Given

- a directed graph $G = (V, E)$,
- two source-sink pairs (s_1, t_1, s_2, t_2),
- edge capacity $\boldsymbol{u} \in \mathbb{R}_{\geq 0}^{|E|}$

Find a pair of **feasible** flow $\boldsymbol{f}_1, \boldsymbol{f}_2 \in \mathbb{R}^{|E|}$ s.t. the sum of the amount of flow routed from s_1 to t_1 and from s_2 to t_2 is maximized.



$maxFlow_1 = 2$
$maxFlow_2 = 2$

2CF as LP

- $(\boldsymbol{f}_1, \boldsymbol{f}_2)$ is a pair of feasible flow if it satisfies

1. **Flow conservation constraints:** for $i \in \{1,2\}$

$$\sum_{incoming} \boldsymbol{f}_i(\cdot, v) - \sum_{outgoing} \boldsymbol{f}_i(v, \cdot) = 0, \qquad \forall v \in V \backslash \{s_i, t_i\}$$

$$\sum_{outgoing} \boldsymbol{f}_i(s, \cdot) = \sum_{incoming} \boldsymbol{f}_i(\cdot, t) = F_i$$

2. **Capacity constraints**

$$\boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) \le \boldsymbol{u}(e), \qquad \forall\, e \in E$$

3. **Direction constraints**

$$\boldsymbol{f}_1(e), \boldsymbol{f}_2(e) \ge 0, \qquad \forall\, e \in E$$

$$\begin{array}{ll} \max & F_1 + F_2 \\ \text{s.t.} & \boldsymbol{B}\boldsymbol{f}_1 = F_1 \boldsymbol{d}_1 \\ & \boldsymbol{B}\boldsymbol{f}_2 = F_2 \boldsymbol{d}_2 \\ & \boldsymbol{f}_1 + \boldsymbol{f}_2 \le \boldsymbol{u} \\ & \boldsymbol{f}_1, \boldsymbol{f}_2 \ge \boldsymbol{0} \end{array}$$

LP solver: $\tilde{O}\left(|E|^{\max\{\omega,\, 2.055\}}\right)$

(since $nnz = \Theta(|E|)$)

1CF solver: $\tilde{O}\left(|E|^{1+o(1)}\right)$

2CF solver: **No faster algorithms known for 2CF beyond general LPs!**

[D.-Kyng-Zhang'22] **LP can be reduced to 2CF!**

Optimization version → Feasibility version

LP

2CF

Optimization version

LP:

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

2CF:

$$\begin{aligned} \max \quad & F_1 + F_2 \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{f}_1 = F_1 \boldsymbol{d}_1 \\ & \boldsymbol{B}\boldsymbol{f}_2 = F_2 \boldsymbol{d}_2 \\ & \boldsymbol{f}_1 + \boldsymbol{f}_2 \le \boldsymbol{u} \\ & \boldsymbol{f}_1, \boldsymbol{f}_2 \ge \boldsymbol{0} \end{aligned}$$

Equivalent hardness

Binary search over K, R^{2cf}

Feasibility version

LP:

$$\begin{aligned} & \boldsymbol{c}^\top \boldsymbol{x} \ge K - \epsilon^{lp} \\ & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} + \epsilon^{lp} \boldsymbol{1} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

$$(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K, \epsilon^{lp})$$

2CF:

$$\begin{aligned} & F_1 + F_2 \ge R^{2cf} - \epsilon^{2cf} \\ & \|\boldsymbol{B}\boldsymbol{f}_1 - F_1 \boldsymbol{d}_1\|_\infty \le \epsilon^{2cf} \\ & \|\boldsymbol{B}\boldsymbol{f}_2 - F_2 \boldsymbol{d}_2\|_\infty \le \epsilon^{2cf} \\ & \boldsymbol{f}_1 + \boldsymbol{f}_2 \le \boldsymbol{u} + \epsilon^{2cf} \boldsymbol{1} \\ & \boldsymbol{f}_1, \boldsymbol{f}_2 \ge \boldsymbol{0} \end{aligned}$$

$$(G = (V, E), \boldsymbol{u}, R^{2cf}, \epsilon^{2cf})$$

Main results

LP

$(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K)$

- $\|\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}\|_{max} \leq X$
- $\|\boldsymbol{x}\|_1 \leq R$
- $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ integer entries

2CF

$(G = (V, E), \boldsymbol{u}, R^{2cf})$

- $|V|, |E| = O(N \log X)$ $\boldsymbol{\tilde{O}(N^2)}$
- $\|\boldsymbol{u}\|_\infty, R^{2cf} = poly(NRX \log X)$

$\boldsymbol{exp(NRX)}$

$\boldsymbol{\tilde{O}(N^2)}$

Reduction algorithm: $O(N \log X)$ →

$N = nnz(\boldsymbol{A})$

Solution mapping algorithm: $O(N \log X)$ ←

$\boldsymbol{x}^{\boldsymbol{lp}}$ (ϵ^{lp})

$\boldsymbol{f}^{\boldsymbol{2cf}}$ (ϵ^{2cf})

$\epsilon^{lp} = poly(NRX \log X)\ \epsilon^{2cf}$
(in exact case, $\epsilon^{lp} = \epsilon^{2cf} = 0$)

Exact case only

- **An immediate implication**

  If we can solve 2CF in $\tilde{O}(|E|^c)$ time, $c \geq 1$, then it can be translated to an LP solver with runtime $\tilde{O}(N^c)$, where $|E| = \tilde{O}(N)$

- **Comparison to [Itai'78]**

  Our proof builds upon Itai's polynomial-time reduction, but we made several improvements

Remark

Our hardness result only rules out possibilities of fast 2CF algorithms for

1. Directed graph: $\boldsymbol{f} \geq \mathbf{0}$
2. Sparse graph: $m = O(n)$, where $|E| = m, |V| = n$
3. High-accuracy regime: polylog dependences on ϵ

Outside of the settings, there indeed exists some fast multi-commodity flow solvers:

	Undirected k-CF	**Directed k-CF**
Low accuracy $(poly(1/\epsilon))$	[She17] $\tilde{O}(mk\epsilon^{-1})$	[Mad10] $\tilde{O}((m+k)n\epsilon^{-2})$
High accuracy $(poly\ \log(1/\epsilon))$	[CY23] $\tilde{O}_{q,p}\left(m^{1+o(1)}k^2\right)$ smoothed k-CF	[BZ23] $\tilde{O}(k^{2.5}\sqrt{m}n^{\omega-1/2})$ faster than LP solver for dense graphs

LP solver for k-CF [CLS19]: $\tilde{O}((mk)^{\omega+o(1)})$

Reduction Algorithm

Overview of reduction algorithm

LP

$(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K)$

$\|\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}\|_{max} \leq X$

$\|\boldsymbol{x}\|_1 \leq R$

Reduction algorithm: $O(N \log X)$

2CF

$(G = (V, E), \boldsymbol{u}, R^{2cf})$

$|V|, |E| = O(N \log X)$

$\|\boldsymbol{u}\|_\infty, R^{2cf} = poly(NRX \log X)$

LEN

2-LEN

1-LEN

FHF

SFF

2CFF

Keep track of:

	LP land	**Flow land**
problem size	$nnz(\boldsymbol{A})$	$\|E\|$
problem magnitude	$\|\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}\|_{max}$	$\|\boldsymbol{u}\|_\infty$

LP land

- Algebra
- Continuous

Flow land

- Graph
- Combinatorics

LP land

LP

$$\boldsymbol{c}^\top \boldsymbol{x} \geq K$$
$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$
$$\boldsymbol{x} \geq 0$$

$O(N)$ ↓ Add slack variables $\alpha, \boldsymbol{s}$

LEN
(Linear Equations with Nonnegative variables)

$$\boldsymbol{c}^\top \boldsymbol{x} - \alpha = K$$
$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{s} = \boldsymbol{b}$$
$$\boldsymbol{x}, \boldsymbol{s}, \alpha \geq 0$$

$O(N \ \log \ X)$ ↓ Bitwise decomposition

2-LEN
$\bar{\boldsymbol{A}}_{\boldsymbol{ij}} \in \{0, \pm 1, \pm 2\}$

$$\bar{\boldsymbol{A}}\bar{\boldsymbol{x}} = \bar{\boldsymbol{b}}$$
$$\bar{\boldsymbol{x}} \geq \boldsymbol{0}$$

$O(N \ \log \ X)$ ↓ Add auxiliary variables $\begin{cases} 2x_i \leftarrow x_i + x_{i'} \\ x_i - x_{i'} = 0 \end{cases}$

1-LEN
$\widehat{\boldsymbol{A}}_{\boldsymbol{ij}} \in \{-1,0,1\}$

$$\widehat{\boldsymbol{A}}\widehat{\boldsymbol{x}} = \widehat{\boldsymbol{b}}$$
$$\widehat{\boldsymbol{x}} \geq \boldsymbol{0}$$

$$\|\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{b}}\|_{max} \leq poly(NRX \log X)$$
$$\|\widehat{\boldsymbol{x}}\|_1 \leq poly(NRX \log X)$$

$$5x_1 + 3x_2 = -1$$

$2^0 + 2^2$ $\quad$ $2^0 + 2^1$ $\quad$ -2^0

↓

$$2^0 \cdot (x_1 + x_2) + 2^1 \cdot x_2 + 2^2 \cdot x_1 = -2^0 \cdot 1$$

$$x_1 + x_2 = -1$$
$$x_2 = 0$$
$$x_1 = 0$$

↓ Add carry terms

$$x_1 + x_2 - 2c_0 = -1$$
$$x_2 + c_0 - 2c_1 = 0$$
$$x_1 + c_1 = 0$$

↓ $c_i \leftarrow d_i - e_i,$ $\; d_i, e_i \geq 0$

$$x_1 + x_2 - 2(d_0 - e_0) = -1$$
$$x_2 + (d_0 - e_0) - 2(d_1 - e_1) = 0$$
$$x_1 + (d_1 - e_1) = 0$$

$$d_i, e_i \leq 2XR$$

LP land → Flow land

1-LEN
$\widehat{A}_{ij} \in \{-1,0,1\}$

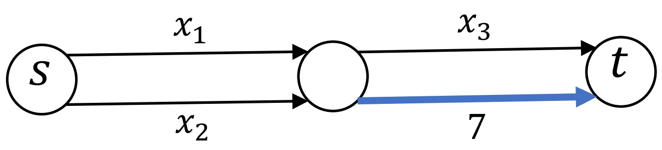
$|E| = O(N \log X)$

$\|\boldsymbol{u}\|_\infty = \|\widehat{\boldsymbol{b}}\|_{max} \leq poly(NRX \log X)$

FHF
(Fixed Homologous Flow problem)

$$x_1 + x_2 - x_3 = 7$$

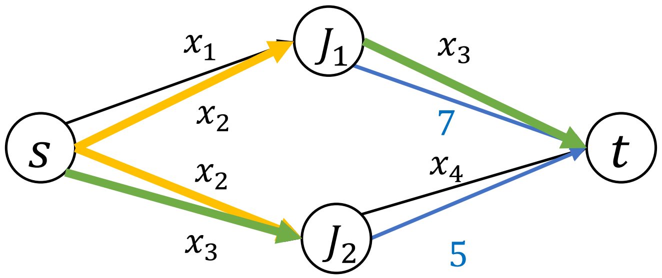
Encode flow conservation constraint



1. **Fixed flow constraint**

$$\boldsymbol{f}(e) = \boldsymbol{u}(e) = RHS$$

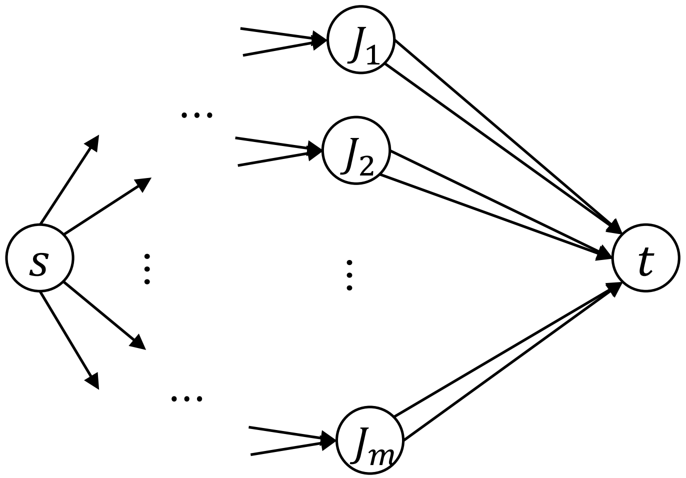
$$x_1 + x_2 - x_3 = 7$$
$$x_2 + x_3 - x_4 = 5$$



2. **Homologous flow constraint**

$$\boldsymbol{f}(e_1) = \cdots = \boldsymbol{f}(e_k)$$

$$\widehat{\boldsymbol{A}}\boldsymbol{x} = \widehat{\boldsymbol{b}}$$



$$|E| = O(nnz(\widehat{\boldsymbol{A}}))$$

$$\|\boldsymbol{u}\|_\infty = \max\left\{\left\|\widehat{\boldsymbol{A}}, \widehat{\boldsymbol{b}}\right\|_{max}, \|\widehat{\boldsymbol{x}}\|_1\right\} \leq poly(NRX \log X)$$

Flow land

Target constraints (2CF):

1. Flow conservation constraints
$$\boldsymbol{B}\boldsymbol{f}_{\boldsymbol{i}} = F_i \boldsymbol{d}_{\boldsymbol{i}}, i \in \{1,2\}$$
2. Capacity constraints
$$\boldsymbol{f}_1 + \boldsymbol{f}_2 \leq \boldsymbol{u}$$
3. Direction constraints: $\boldsymbol{f}_1, \boldsymbol{f}_2 \geq \boldsymbol{0}$

drop extra constraints step by step

FHF constrains:

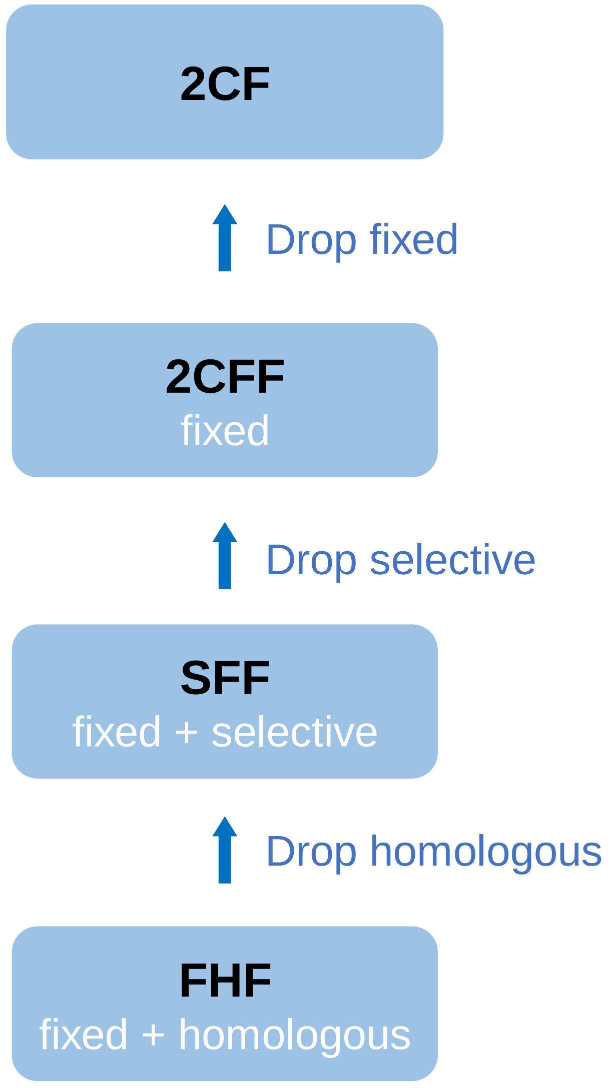
- Target constraints 1, 2, 3

4. Fixed flow constraints
$$\boldsymbol{f}(e) = \boldsymbol{u}(e)$$
5. Homologous flow constraints
$$\boldsymbol{f}(e_1) = \cdots = \boldsymbol{f}(e_k)$$

Introduce

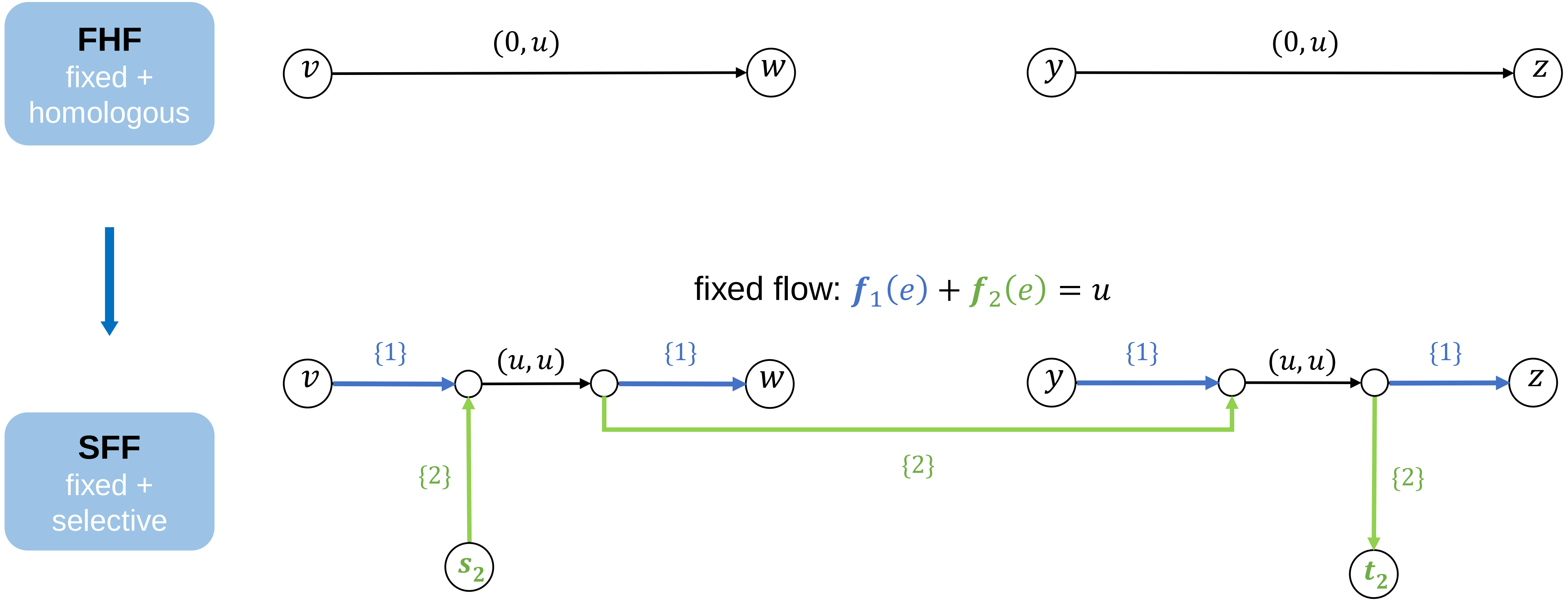
- 2^{nd} commodity
- Selective flow constraints

$$\boldsymbol{f}_{\bar{\imath}}(e) = 0, \qquad \bar{\imath} \neq i$$



Simplified reduction chain. More steps needed in the complete version.

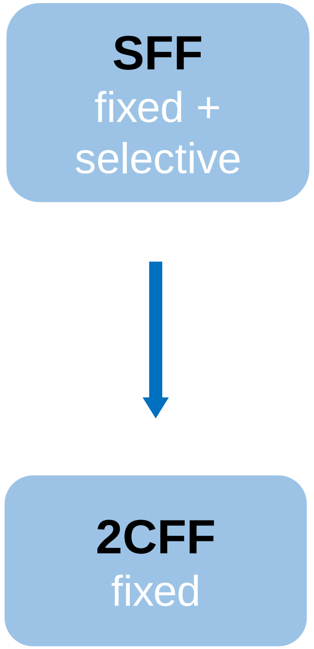
Drop homologous

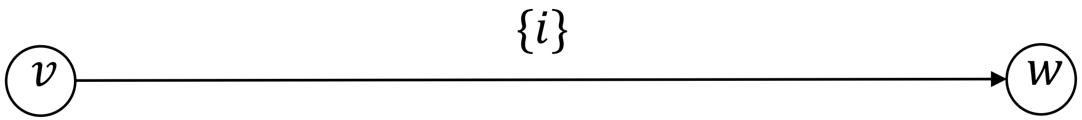


selective for commodity 1: $f_2(e) = 0$

selective for commodity 2: $f_1(e) = 0$

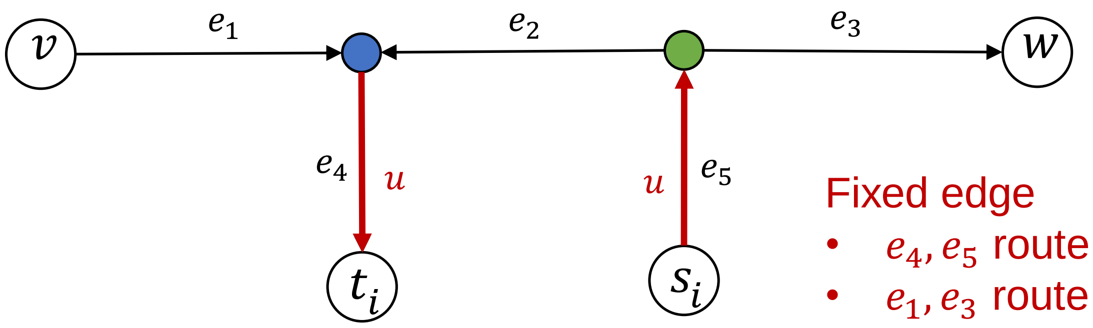
Drop selective





Only outgoing edge is e_4
- e_4 only allow f_i
- e_1 only routes f_i

Only incoming edge is e_5
- e_5 only allow f_i
- e_3 only routes f_i



Fixed edge
- e_4, e_5 route the same
- e_1, e_3 route the same

- Only fixed flow constraint remains
- High-level idea of dropping fixed flow constraint (needs several steps)
  - Fixed flow edge = saturated edge
  - In feasibility 2CF, with a carefully chosen R^{2cf}, use $F_1 + F_2 \geq R^{2cf}$ to force fixed flow edges to get saturated automatically

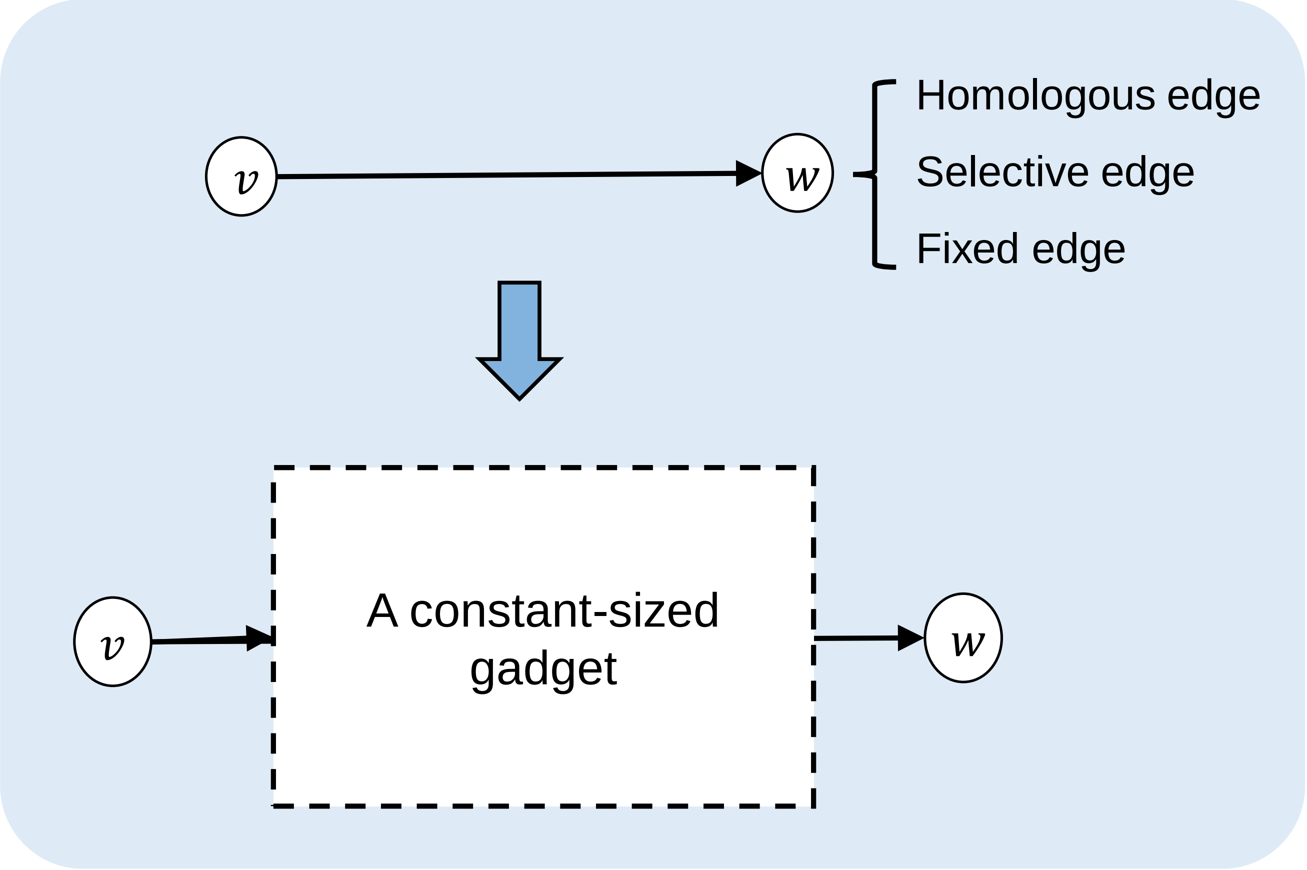
Flow Land

1-LEN
$\widehat{A}_{ij} \in \{-1,0,1\}$

$|E| = O(N \log X)$

$\|\boldsymbol{u}\|_\infty = \|\widehat{\boldsymbol{b}}\|_{max} \leq poly(NRX \log X)$

FHF
(Fixed Homologous Flow problem)



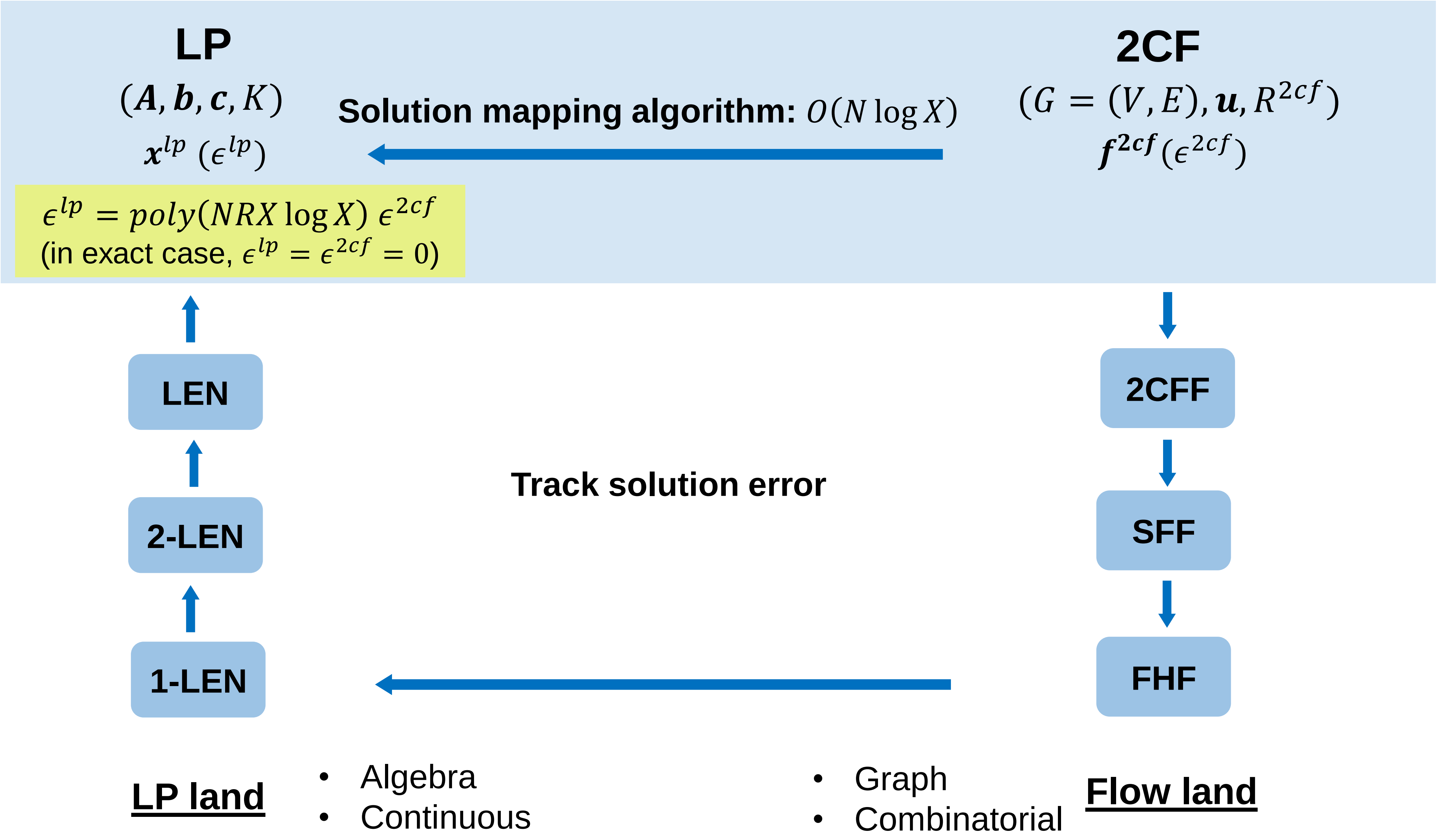
$|E| = O(N \log X)$

$\|\boldsymbol{u}\|_\infty \leq poly(NRX \log X)$

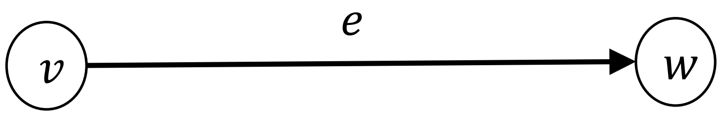
2CF

Solution Mapping & Error Analysis

Overview of solution mapping



Flow land



Instance $\boldsymbol{Q}$: $(\boldsymbol{f}^{\boldsymbol{Q}}, \boldsymbol{\epsilon}^{\boldsymbol{Q}})$

Instance $\boldsymbol{P}$: $(\boldsymbol{f}^{\boldsymbol{P}}, \boldsymbol{\epsilon}^{\boldsymbol{P}})$

Mapping rule: $\boldsymbol{f}^P(e) \leftarrow \boldsymbol{f}^Q(e_1)$

Error analysis

$$\epsilon^P = O(|E^P|)\ \epsilon^Q$$

Intuition: we map the flow of a gadget to a single edge

- error accumulates in an **additive manner**

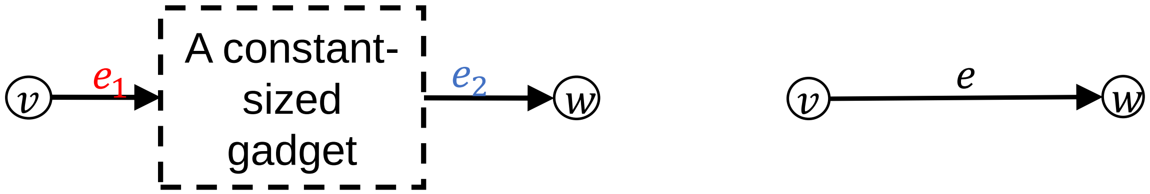
  → **per gadget** error blows up by a constant

- At most $|E^P|$ gadgets

  → the **total** error blows up by $O(|E^P|)$

Solution Mapping

Flow land



Instance Q: (f^Q, ϵ^Q) → **Instance P**: (f^P, ϵ^P)

Mapping rule: $f^P(e) \leftarrow f^Q(e_1)$

Error analysis

$$\epsilon^P = O(|E^P|)\ \epsilon^Q$$

Intuition: we map the flow of a gadget to a single edge

- error accumulates in an **additive manner**

  → error blows up by a constant per gadget

- At most $|E^P|$ gadgets

  → the total error blows up by $O(|E^P|)$

LP land

$$x^Q = \begin{pmatrix} x^P \\ x^{new} \end{pmatrix} \qquad x^P$$

Instance Q: (x^Q, ϵ^Q) → **Instance P**: (x^P, ϵ^P)

Direct mapping

Error analysis

$$\epsilon^P = \text{poly}\,(N^P, X^P) \cdot \epsilon^Q$$

Simple algebra

Summary

$$\begin{aligned} \max \quad & F_1 + F_2 \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{f}_1 = F_1\boldsymbol{d}_1 \\ & \boldsymbol{B}\boldsymbol{f}_2 = F_2\boldsymbol{d}_2 \\ & \boldsymbol{f}_1 + \boldsymbol{f}_2 \le \boldsymbol{u} \\ & \boldsymbol{f}_1, \boldsymbol{f}_2 \ge \boldsymbol{0} \end{aligned}$$

$(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K)$

- $\|\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}\|_{max} \le X$
- $\|\boldsymbol{x}\|_1 \le R$
- $\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}$ integer entries

→

$(G = (V, E), \boldsymbol{u}, R^{2cf})$

- $|V|, |E| = O(N \log X)$
- $\|\boldsymbol{u}\|_\infty, R^{2cf} = poly(NRX \log X)$
- $\epsilon^{2cf} = \frac{\epsilon^{lp}}{poly(NRX \log X)}$

2CF $\le$ LP

LP $\le$ 2CF

2CF $=$ LP

Summary & open problems

LP

$(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}, K),\ \boldsymbol{x}^{lp}(\epsilon^{lp})$

$$\begin{cases} \|\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}\|_{max} \leq X \\ \|\boldsymbol{x}\|_1 \leq R \\ \boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c} \text{ integer entries} \end{cases}$$

Reduction: $O(N \log X)$

Solution mapping: $O(N \log X)$

2CF

$(G = (V, E), \boldsymbol{u}, R^{2cf}),\ \boldsymbol{f}^{2cf}(\epsilon^{2cf})$

$$\begin{cases} |V|, |E| = O(N \log X) \\ \|\boldsymbol{u}\|_\infty, R^{2cf} = poly(NRX \log X) \\ \epsilon^{2cf} = \dfrac{\epsilon^{lp}}{poly(NRX \log X)} \end{cases}$$

Q1. Exist an efficient reduction for non poly-bounded LP?

Q3. We have dense LP -> sparse 2CF,

what about dense LP -> dense 2CF?

ϵ^{lp} **LEN** $O(N)$

Q2. Exist a strongly polynomial reduction?

$\Omega\left(\frac{\epsilon^{lp}}{X}\right)$ **2-LEN** $O(N \log X)$

$\Omega\left(\frac{\epsilon^{lp}}{(N \log X)\, X}\right)$ **1-LEN** $O(N \log X)$

$O(N \log X)$ **2CFF** $\Omega\left(\frac{\epsilon^{lp}}{(N \log X)^5\, NRX^3}\right)$

$O(N \log X)$ **SFF** $\Omega\left(\frac{\epsilon^{lp}}{(N \log X)^4\, NRX^3}\right)$

$O(N \log X)$ **FHF** $\Omega\left(\frac{\epsilon^{lp}}{(N \log X)^2\, NRX^3}\right)$

Thank you !

Backup slides

Hardness of two-commodity Laplacians

- State-of-the-art faster algorithms (LPs, maxflow) are **Interior-Point-Method-based** algorithms
  - Interior Point Methods (IPM): ways of solving convex optimization problems via a sequence of linear systems
- **Laplacians** linear system: the linear equations that arise in IPM for maxflow
- **2-commodity Laplacians** linear system: the linear equations that arise in IPM for 2CF

Theorem [Kyng-Zhang'17]
If we can quickly solve linear systems in **2-commodity Laplacians**, then we can quickly solve linear systems in any matrix.

Implications:

- Faster 2CF algorithms were unlikely to be IPM-based
- However, it remains open that if other families of algorithms could succeed

Integral entry assumption is mild

$(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is polynomially-bounded

- Magnitude $\|\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}\|_{max} \leq X$
- Polytope radius $\|\boldsymbol{x}\|_1 \leq R$

$$\begin{aligned} \max \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

We round by at most $\frac{\epsilon}{3R}$

- entries of $\boldsymbol{A}$ down to $\widetilde{\boldsymbol{A}}$
- entries of $\boldsymbol{b}, \boldsymbol{c}$ up to $\widetilde{\boldsymbol{b}}, \widetilde{\boldsymbol{c}}$

If $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is feasible, then $(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{b}}, \widetilde{\boldsymbol{c}})$ is feasible

Since entries of $(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{b}}, \widetilde{\boldsymbol{c}})$ has a **logarithmic number of bits**, we can **scale** all of them to **polynomially-bounded integers**

If $\widetilde{\boldsymbol{x}}$ is a solution to $(\widetilde{\boldsymbol{A}}, \widetilde{\boldsymbol{b}}, \widetilde{\boldsymbol{c}})$ with $\epsilon/3$ additive error.

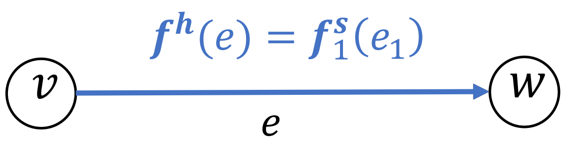
Then $\widetilde{\boldsymbol{x}}$ is also a solution to $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ with ϵ additive error.

$$\boldsymbol{A}\,\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{x}} + (\boldsymbol{A} - \widetilde{\boldsymbol{A}})\widetilde{\boldsymbol{x}} \leq \left(\widetilde{\boldsymbol{b}} + \frac{\epsilon}{3}\boldsymbol{1}\right) + \frac{\epsilon}{3R} \cdot R = \boldsymbol{b} + \epsilon\boldsymbol{1}$$

$$\boldsymbol{c}^\top\,\widetilde{\boldsymbol{x}} = \widetilde{\boldsymbol{c}}^\top\,\widetilde{\boldsymbol{x}} + (\boldsymbol{c} - \widetilde{\boldsymbol{c}})^\top\widetilde{\boldsymbol{x}} \geq \left(\widetilde{\boldsymbol{c}}^\top\widetilde{\boldsymbol{x}}^* - \frac{\epsilon}{3}\right) - \frac{\epsilon}{3R} \cdot R = \widetilde{\boldsymbol{c}}^\top\widetilde{\boldsymbol{x}}^* - \frac{2}{3}\epsilon \geq \boldsymbol{c}^\top\boldsymbol{x}^* - \frac{2}{3}\epsilon$$

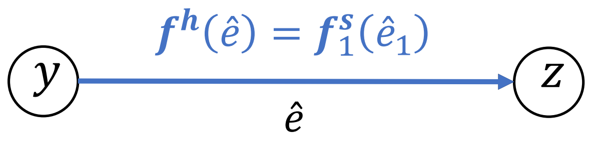
SFF → FHF

FHF
$(\boldsymbol{f}^{\boldsymbol{h}}, \epsilon^h)$



- Error in congestion: $\leq \epsilon^s$
- Error in demand: $\leq O(|E^h|)\epsilon^s$
- Error in homology:

$$\left|\boldsymbol{f}^h(e) - \boldsymbol{f}^h(\hat{e})\right| \leq \Theta(\epsilon^s)$$

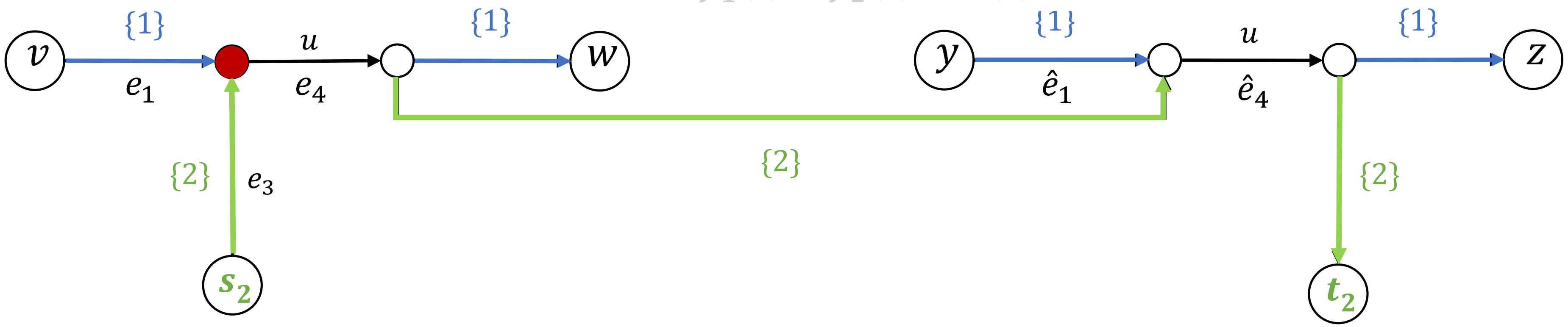


Goal: after mapping $\boldsymbol{f}^s$ with error ϵ^s, how to bound ϵ^h of $\boldsymbol{f}^h$?

Error in demand: for $i \in \{1,2\}$
$|\boldsymbol{f}_i^s(e_1) + \boldsymbol{f}_i^s(e_3) - \boldsymbol{f}_i^s(e_4)| \leq \epsilon^s$

Error in congestion (fixed):
$u - \epsilon^s \leq \boldsymbol{f}_1^s(\hat{e}_4) + \boldsymbol{f}_2^s(\hat{e}_4) \leq u + \epsilon^s$

~~fixed flow: $\boldsymbol{f}_1(e) + \boldsymbol{f}_2(e) = \boldsymbol{u}(e)$~~



SFF
$(\boldsymbol{f}^s, \epsilon^s)$

~~selective for commodity 1: $\boldsymbol{f}_2(e) = 0$~~

Error in type for $\{1\}$: $\boldsymbol{f}_2^s(e_1) \leq \epsilon^s$

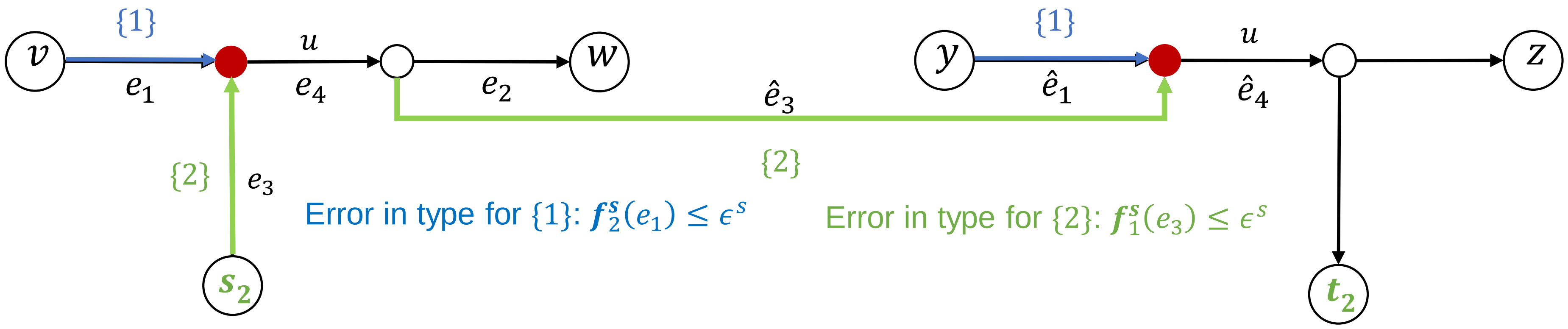
~~selective for commodity 2: $\boldsymbol{f}_1(e) = 0$~~

Error in type for $\{2\}$: $\boldsymbol{f}_1^s(e_3) \leq \epsilon^s$

Error in homology (SFF → FHF)

Error in demand: for $i \in \{1,2\}$
$|\boldsymbol{f}_i^s(e_1) + \boldsymbol{f}_i^s(e_3) - \boldsymbol{f}_i^s(e_4)| \leq \epsilon^s$

Error in congestion (fixed):
$u - \epsilon^s \leq \boldsymbol{f}_1^s(\hat{e}_4) + \boldsymbol{f}_2^s(\hat{e}_4) \leq u + \epsilon^s$



$\left|\boldsymbol{f}^h(e) - \boldsymbol{f}^h(\hat{e})\right| = |\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(\hat{e}_1)|$ By solution mapping

$\leq \left|\left(\boldsymbol{f}_1^s(e_1) + \boldsymbol{f}_1^s(e_3)\right) - \left(\boldsymbol{f}_1^s(\hat{e}_1) + \boldsymbol{f}_1^s(\hat{e}_3)\right)\right| + |\boldsymbol{f}_1^s(e_3) - \boldsymbol{f}_1^s(\hat{e}_3)|$ $\leq \Theta(\epsilon^s)$

Constant sized gadget

Error in demand of red vertices:
$|\boldsymbol{f}^s(e_1) + \boldsymbol{f}^s(e_3) - \boldsymbol{f}^s(e_4)| \leq 2\epsilon^s$
$|\boldsymbol{f}^s(\hat{e}_1) + \boldsymbol{f}^s(\hat{e}_3) - \boldsymbol{f}^s(\hat{e}_4)| \leq 2\epsilon^s$

Error in type for e_3 and $\hat{e}_3$:
$|\boldsymbol{f}_1^s(e_3) - \boldsymbol{f}_1^s(\hat{e}_3)| \leq \epsilon^s$

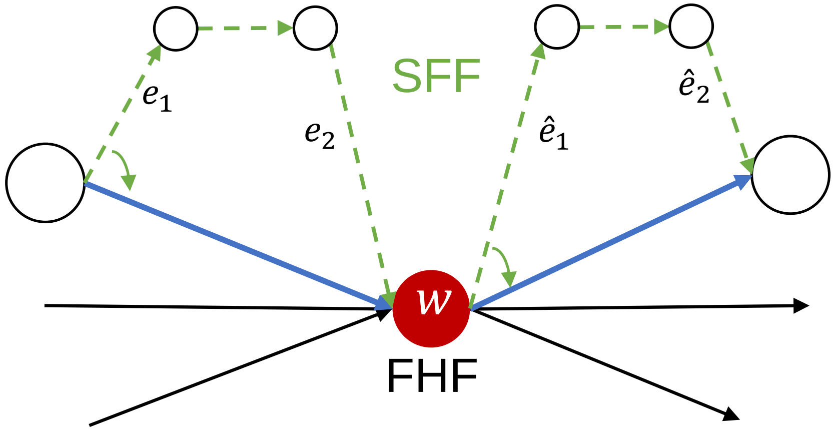
$\boldsymbol{f}^s(e) = \boldsymbol{f}_1^s(e) + \boldsymbol{f}_2^s(e)$

Error in type for e_1 and $\hat{e}_1$:
$|\boldsymbol{f}_2^s(e_1) - \boldsymbol{f}_2^s(\hat{e}_1)| \leq \epsilon^s$

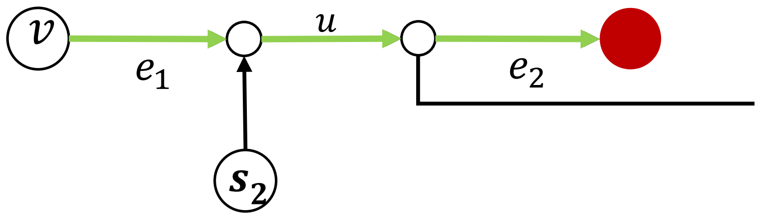
Error in congestion of e_4 and $\hat{e}_4$:
$|\boldsymbol{f}^s(\hat{e}_4) - \boldsymbol{f}^s(e_4)| \leq 2\epsilon^s$

Error in type for e_3 and $\hat{e}_3$:
$|\boldsymbol{f}_1^s(e_3) - \boldsymbol{f}_1^s(\hat{e}_3)| \leq \epsilon^s$

Error in demand (SFF → FHF)



H: set of homologous edges



$$\max_{w \in V^h \setminus \{s,t\}} \left| \sum_{incoming} \boldsymbol{f}^h(\cdot, w) - \sum_{outgoing} \boldsymbol{f}^h(w, \cdot) \right|$$

Solution mapping $\quad \sum_{in \in H} f_1^s(e_1) + \sum_{in \notin H} \boldsymbol{f}_1^s(\cdot, w) \qquad \sum_{out \in H} \boldsymbol{f}_1^s(\hat{e}_1) + \sum_{out \notin H} \boldsymbol{f}_1^s(w, \cdot)$

Error in demand of w in SFF $\quad \epsilon^s + \left(\sum_{in \in H} \boldsymbol{f}_1^s(e_2) + \sum_{in \notin H} \boldsymbol{f}_1^s(\cdot, w) \right)$

$$= \max_{w \in V^h \setminus \{s,t\}} \epsilon^s + \left| \sum_{in \in H} \boldsymbol{f}_1^s(e_1) - \sum_{in \in H} \boldsymbol{f}_1^s(e_2) \right|$$

$$\leq \max_{w \in V^h \setminus \{s,t\}} \epsilon^s + \sum_{in \in H} |\boldsymbol{f}_1^s(e_1) - \boldsymbol{f}_1^s(e_2)|$$

$\leq \Theta(\epsilon^s)$ by constant sized gadget

$$\leq \Theta(|E^h|) \cdot \epsilon^s$$

by at most $|E^h|$ gadgets